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TERMINOLOGY

chain rule
composite function
Euler's number
exponential decay
exponential growth
product rule
quotient rule

FURTHER DIFFERENTIATION AND APPLICATIONS

DERIVATIVES, EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

Prior learning

EXPONENTIAL FUNCTIONS

- estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$ using technology, for various values of $a > 0$ (ACMMM098)
- recognise that e is the unique number a for which the above limit is 1 (ACMMM099)
- establish and use the formula $\frac{d}{dx}(e^x) = e^x$ (ACMMM100)
- use exponential functions and their derivatives to solve practical problems (ACMMM101)

TRIGONOMETRIC FUNCTIONS

- establish the formulas $\frac{d}{dx}(\sin x) = \cos x$, and $\frac{d}{dx}(\cos x) = -\sin x$ by numerical estimations of the limits and informal proofs based on geometric constructions (ACMMM102)
- use trigonometric functions and their derivatives to solve practical problems (ACMMM103)

DIFFERENTIATION RULES

- understand and use the product and quotient rules (ACMMM104)
- understand the notion of composition of functions and use the chain rule for determining the derivatives of composite functions (ACMMM105)
- apply the product, quotient and chain rule to differentiate functions such as xe^x, $\tan x$, $\frac{1}{x^n}$, $x \sin x$, $e^{-x} \sin x$ and $f(ax + b)$ (ACMMM106) AC

1.01 THE PRODUCT AND QUOTIENT RULES

In Year 11 you learnt how to differentiate, using first principles and shorter rules.

IMPORTANT

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\frac{d}{dx}(x^n) = nx^{n-1}$ for $n \in \mathbf{N}$

$\frac{d}{dx}(k) = 0$ where k is a constant

$\frac{d}{dx}(kx^n) = knx^{n-1}$ where k is a constant

$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

The derivative $\frac{dy}{dx}$ is the gradient function of the curve $y = f(x)$.

A linear function through (x_1, y_1) with gradient m is given by $y - y_1 = m(x - x_1)$.

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The formula $y - y_1 = m(x - x_1)$ can be used to find the equation of a tangent at a point, as the gradient is given by $m = \frac{dy}{dx}$.

WS Basic differentiation

In this section you will learn some more differentiation rules. The derivative of a product of functions $f(x)\,g(x)$ can be found as follows.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x+h)+f(x)[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x+h)}{h}+\frac{f(x)[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\times\lim_{h\to 0}g(x+h)+\lim_{h\to 0}f(x)\times\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}$$

It is easier to remember the **product rule** by using $u = f(x)$ and $v = g(x)$ and writing $v' = \frac{dv}{dx}$ and $u' = \frac{du}{dx}$.

IMPORTANT

The **product rule** for functions u and v can be written as $\frac{d}{dx}(uv) = u'v + uv'$.

The example below shows how you can use the product rule to avoid expanding brackets.

Example 1

a Find the derivative of $3x^2(x - 5)$.

b For the function $f(x) = (3x^4 - 2x + 7)(x^5 + 2x^2 + x - 1)$, find $f'(-2)$.

Solution

a Write the function as a product.

Let $y = 3x^2(x - 5) = uv$, where $u = 3x^2$ and $v = x - 5$

Find the derivatives.

$u' = 6x$ and $v' = 1$

Use $\frac{d}{dx}(uv) = u'v + uv'$.

$$\frac{dy}{dx} = 6x\times(x-5)+3x^2\times 1$$

Simplify.

$$\begin{aligned} &= 6x^2 - 30x + 3x^2 \\ &= 9x^2 - 30x \end{aligned}$$

Write the answer.

$$\frac{d}{dx}[3x^2(x-5)] = 9x^2 - 30x$$

b	Write the function as a product.	Let $f(x) = uv$, where $u = 3x^4 - 2x + 7$ and $v = x^5 + 2x^2 + x - 1$
	Differentiate.	$u' = 12x^3 - 2$ and $v' = 5x^4 + 4x + 1$
	Write the product rule.	$f'(x) = u'v + uv'$
	Substitute the functions.	$= (12x^3 - 2)(x^5 + 2x^2 + x - 1) + (3x^4 - 2x + 7)(5x^4 + 4x + 1)$
	Substitute the value.	$f'(-2) = [12(-2)^3 - 2][(-2)^5 + 2(-2)^2 + (-2) - 1] + [3(-2)^4 - 2(-2) + 7][5(-2)^4 + 4(-2) + 1]$
	Simplify.	$= 6953$

In part **b** above, you could simplify the derivative by multiplying out the brackets, but that wouldn't have made it any quicker to find $f'(-2)$.

WS
The product rule

You can work out the derivative of $f(x) = \dfrac{u(x)}{v(x)}$ as follows.

$$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Now $f(x+h) - f(x) = \dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}$

$$= \frac{u(x+h)v(x) - u(x)v(x+h)}{v(x+h)v(x)}$$

So
$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{u(x+h)v(x) - u(x)v(x+h)}{v(x+h)v(x)h}\\
&= \lim_{h\to 0}\frac{u(x+h)v(x) - u(x)v(x+h)}{h} \times \lim_{h\to 0}\frac{1}{v(x+h)v(x)}\\
&= \left[\lim_{h\to 0}\frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{h}\right] \times \frac{1}{v(x)v(x)}\\
&= \frac{1}{[v(x)]^2} \times \lim_{h\to 0}\frac{[u(x+h) - u(x)]v(x) - u(x)[v(x+h) - v(x)]}{h}\\
&= \frac{1}{[v(x)]^2} \times \left[\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} \times v(x) - u(x) \times \lim_{h\to 0}\frac{v(x+h) - v(x)}{h}\right]\\
&= \frac{1}{[v(x)]^2}[u'(x)v(x) - u(x)v'(x)]\\
&= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}
\end{aligned}$$

This is very similar to the product rule.

IMPORTANT

The **quotient rule** for functions u and v can be written as $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{u'v - uv'}{v^2}$.

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Example 2

Find the derivative of $\dfrac{3x-5}{5x+2}$.

Solution

Write as a quotient. Let $y=\dfrac{u}{v}$, where $u=3x-5$ and $v=5x+2$.

Find the derivatives. $u'=3$ and $v'=5$

Write the quotient rule. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{u'v-uv'}{v^2}$

Substitute the functions. $=\dfrac{3\times(5x+2)-(3x-5)\times 5}{(5x+2)^2}$

Multiply out the brackets. $=\dfrac{15x+6-15x+25}{(5x+2)^2}$

Simplify and write the answer. $\dfrac{d}{dx}\left[\dfrac{3x-5}{5x+2}\right]=\dfrac{31}{(5x+2)^2}$

Sometimes you will need to differentiate functions with negative or fractional powers. Remember these definitions.

IMPORTANT

$$x^{-n}=\frac{1}{x^n}$$

$$x^{\frac{1}{n}}=\sqrt[n]{x}$$

$$x^{\frac{m}{n}}=\sqrt[n]{x^m}=\left(\sqrt[n]{x}\right)^m$$

WS The quotient rule

You can find the derivative of x^{-n} using the quotient rule.

Write as a quotient. $x^{-n}=\dfrac{1}{x^n}=\dfrac{u}{v}$, where $u=1$ and $v=x^n$

Find the derivatives. $u'=0$ and $v'=nx^{n-1}$

Write the quotient rule. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{u'v-uv'}{v^2}$

Substitute the functions. $\dfrac{d}{dx}(x^{-n})=\dfrac{0\times x^n-nx^{n-1}\times 1}{(x^n)^2}$

Simplify. $=\dfrac{-nx^{n-1}}{x^{2n}}$

Express as a single power. $=-nx^{-n-1}$

So the derivative of x^{-n} is $-nx^{-n-1}$. This means that the general rule $\dfrac{d}{dx}(x^n)=nx^{n-1}$ works when n is *any* integer (both positive and negative numbers).

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This rule actually works for any real numbers, although the proof is beyond the scope of this course. You can see it for fractions like the derivative of $\sqrt{x} = x^{\frac{1}{2}}$.

It is easiest if you change $\sqrt{x+h} - \sqrt{x}$ to another form using the difference of squares.

$$\begin{aligned}
\sqrt{x+h} - \sqrt{x} &= \left(\sqrt{x+h} - \sqrt{x}\right) \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{x+h-x}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{h}{\sqrt{x+h} + \sqrt{x}}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d}{dx}\left(x^{\frac{1}{2}}\right) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \times \frac{h}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}} \\
&= \frac{1}{2} x^{-\frac{1}{2}}
\end{aligned}$$

So the derivative of $x^{\frac{1}{2}}$ is $\frac{1}{2} x^{\frac{1}{2}-1}$, which is the same rule again.

Example 3

Find the derivatives of each of the following.

a x^{-7}

b $x^{\frac{1}{5}}$.

c For $f(x) = \frac{1}{x^2}$, find $f'(5)$.

d If $g(x) = \sqrt[3]{x}$, find $g(64)$.

Solution

a Use $\frac{d}{dx}(x^n) = nx^{n-1}$. $\qquad \frac{d}{dx} x^{-7} = -7x^{-7-1}$

Simplify. $\qquad = -7x^{-8}$

b Use $\frac{d}{dx}(x^n) = nx^{n-1}$. $\qquad \frac{d}{dx}\left(x^{\frac{1}{5}}\right) = \frac{1}{5} x^{\frac{1}{5}-1} = \frac{1}{5} x^{-\frac{4}{5}}$

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c Change $\frac{1}{x^2}$ to index form. $\quad f(x)=\frac{1}{x^2}=x^{-2}$

Now use $\frac{d}{dx}(x^n)=nx^{n-1}$. $\quad \frac{df}{dx}=-2x^{-2-1}$

$$=-2x^{-3}$$

Substitute the value. $\quad f'(5)=-2\times 5^{-3}$

$$=-\frac{2}{125}$$

d Change $\sqrt[3]{x}$ into index form. $\quad g(x)=\sqrt[3]{x}=x^{\frac{1}{3}}$

Differentiate using $\frac{d}{dx}(x^n)=nx^{n-1}$. $\quad g'(x)=\frac{1}{3}x^{\frac{1}{3}-1}=\frac{1}{3}x^{-\frac{2}{3}}$

Substitute the value. $\quad g'(64)=\frac{1}{3}\times 64^{-\frac{2}{3}}$

Write without the fractional index. $\quad =\frac{1}{3}\times\frac{1}{64^{\frac{2}{3}}}$

Write in surd form. $\quad =\frac{1}{3\left(\sqrt[3]{64}\right)^2}$

Simplify. $\quad =\frac{1}{48}$

EXERCISE 1.01 The product and quotient rules

Concepts and techniques

1 Example 1 Differentiate the following.

a $x^3(2x+3)$ b $(3x-2)(2x+1)$ c $3x(5x+7)$
d $4x^4(3x^2-1)$ e $2x(3x^4-x)$

2 Find the derivative of

a $(3x-7)(2x+5)$ b $(5x+2)(x^2-1)$ c $(5x-3)(x^3+2x-1)$
d $(x^2+7)(x^2-x-1)$ e $(x^3+3)(x^4+2x^3-5x^2+x-2)$

3 Example 2 Find the derivatives of the following.

a $\frac{1}{2x-1}$ b $\frac{3x}{x+5}$ c $\frac{x^3}{x^2-4}$ d $\frac{x-3}{5x+1}$
e $\frac{x-7}{x^2}$ f $\frac{5x+4}{x+3}$ g $\frac{x}{2x^2-x}$ h $\frac{x+4}{x-2}$
i $\frac{2x+7}{4x-3}$ j $\frac{x+5}{3x+1}$

4 Example 3 Find the derivatives of the following.

a x^{-4} b x^{-8} c $2x^{-3}$ d $5x^{-11}$ e $\frac{x^{-9}}{5}$

f $x^{\frac{1}{2}}$ g $x^{\frac{1}{4}}$ h $3x^{\frac{1}{7}}$ i $5x^{\frac{2}{3}}$ j $2x^{-\frac{1}{2}}$

5 Differentiate

a $\frac{1}{x^5}$ b $\frac{1}{x^6}$ c $\frac{2}{x^3}$ d $\frac{1}{3x^2}$ e $\frac{4}{5x}$

f $\sqrt{x}$ g $\sqrt[6]{x}$ h $4\sqrt[3]{x}$ i $\sqrt[3]{x^2}$ j $\sqrt{x^5}$

6 Find each of the following.

a $f'(4)$ if $f(x) = \frac{1}{x}$ b $g'(-2)$ if $g(x) = \frac{x^2-5}{x+3}$

c $\left.\frac{dy}{dx}\right|_{x=3}$ if $y = (2x^2 + 3x - 5)(x^3 - x^2 + 8)$ d $p'(8)$ if $p(t) = 4t^{\frac{5}{3}} - t^{-\frac{4}{3}}$

e $h'(\frac{1}{2})$ for $h(y) = y^{-5}$

7 Find the gradient of the tangent to the curve

a $y = x^2(3x + 2)$ at the point where $x = 4$

b $y = \frac{1}{x}$ where $x = 3$

c $y = \frac{2x+1}{x-2}$ at the point $(1, -3)$

Reasoning and communication

8 For what values of x is the derivative of each of the following positive?

a $\frac{1}{x^2}$ b $x - \sqrt[3]{x}$ c $-6x^{-4}$ d $\frac{x+3}{x^2-5}$ e x^{-3}

9 The curve $y = x^2(3x - 2)$ has two tangents with gradient 5 at points M and N. Find the coordinates of these points.

10 Find any x-values on the curve $y = \frac{2x-1}{x+3}$, where the gradient of the tangent is $\frac{7}{25}$.

11 Find the equation of the tangent to the curve $y = \frac{x^2-1}{x+3}$ at the point where $x = 2$.

12 In economics, the marginal cost of production is the additional cost of a small increase in production. It is the rate of change of the cost function (i.e. cost per item) as production increases. The cost function for the production of engines is given by

$$C(x) = (x + 1)(0.04x^2 - 10x + 20)^2$$

where the cost $C(x)$ of producing x engines is given in dollars.

a Find the cost of producing 20 engines.

b Find the marginal cost if

i 20 engines are made ii 50 engines are made.

13 a Show that $\left(x^{\frac{1}{4}} - y^{\frac{1}{4}}\right)\left(x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} + y^{\frac{3}{4}}\right) = x - y$.

b Hence show that $\frac{d}{dx}x^{\frac{1}{4}} = -\frac{1}{4}x^{-\frac{3}{4}}$.

14 Show that $(x - y)(x^2 + xy + y^2) = x^3 - y^3$ and hence that the derivative of $\sqrt[3]{x}$ is $\frac{1}{3\sqrt[3]{x^2}}$.

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1.02 THE CHAIN RULE

A **composite function** is a function of another function, like $f[g(x)]$. For example, if $f(x) = x^3$ and $g(x) = x^5$, then the composite function $f[g(x)] = f(x^5) = (x^5)^3 = x^{15}$, where $g = x^5$.

Write $h(x) = x^{15}$. Then $h(x) = f[g(x)]$ and clearly $h'(x) = 15x^{14}$.

Now $f'(x) = 3x^2$, so $f'(g) = 3(x^5)^2 = 3x^{10}$

and $g'(x) = 5x^4$

To get $h'(x) = 15x^{14}$, from $f'(g)$ and $g'(x)$ you have to multiply

$$15x^{14} = 3x^{10} \times 5x^4 = f'(g) \times g'(x)$$

This applies to *all* composite functions.

IMPORTANT

The **chain rule** states that $\frac{d}{dx}\{f[g(x)]\} = f'(g)g'(x)$, where $g = g(x)$.

You can prove this is true for all composite functions as follows.

$$\begin{aligned}
\frac{d}{dx}f[g(x)] &= \lim_{h\to 0}\frac{f[g(x+h)]-f[g(x)]}{h} \\
&= \lim_{h\to 0}\left(\frac{f[g(x+h)]-f[g(x)]}{h}\times\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right) \\
&= \lim_{h\to 0}\left(\frac{f[g(x+h)]-f[g(x)]}{g(x+h)-g(x)}\times\frac{g(x+h)-g(x)}{h}\right) \\
&= \lim_{h\to 0}\frac{f[g(x+h)]-f[g(x)]}{g(x+h)-g(x)}\times\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}
\end{aligned}$$

Now write $H = g(x + h) - g(x)$.

Then $g(x + h) = g(x) + H$.

As $h \to 0$, $g(x + h) \to g(x)$, so $g(x + h) - g(x) \to 0$, so $H \to 0$.

Substitution gives

$$\begin{aligned}
\frac{d}{dx}f[g(x)] &= \lim_{H\to 0}\frac{f(g+H)-f(g)}{H}\times\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(g)\times g'(x)
\end{aligned}$$

There are other ways to write the chain rule, and some teachers and students prefer one of these forms.

IMPORTANT

The **chain rule** can also be written as:

If $f(x) = v(u(x))$, then $f'(x) = v'(u)u(x)$, or

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

Example 4

a Find the derivatives of

i $(x^3 - 1)^5$ ii $y = (2x^3 + x + 7)^2$ iii $f(x) = 3(5x^2 - 4x + 2)^8$

b Find the equation of the tangent to the curve $y = (2x - 1)^6$ at the point $(1, 1)$.

Solution

a i Write as a function of a function. Let $y = f[g(x)]$, where $f(x) = x^5$ and $g(x) = x^3 - 1$

Find the derivatives. $f'(x) = 5x^4$ and $g'(x) = 3x^2$

Write the chain rule. $\frac{dy}{dx} = f'(g) \times g'(x)$

Substitute the derivatives. $= 5g^4 \times 3x^2$

Now substitute $g(x)$. $= 5(x^3 - 1)^4 \times 3x^2$

Write in the usual way. $= 15x^2(x^3 - 1)^4$

ii Write as a function of a function. Let $y = y[u(x)]$, where $y(u) = u^2$ and $u(x) = 2x^3 + x + 7$

Find the derivatives. $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 3x^2 + 1$

Write the rule. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Substitute the derivatives. $= 2u \times (3x^2 + 1)$

Now substitute u. $= 2(2x^3 + x + 7)\ (6x^2 + 1)$

iii Write as a function of a function. Let $f(x) = v[u(x)]$, where $v(u) = 3u^8$ and $u(x) = 5x^2 - 4x + 2$

Find the derivatives. $v'(u) = 24u^7$ and $u(x) = 10x - 4$

Write the chain rule. $f'(x) = v'(u)u'(x)$

Substitute the derivatives. $= 24u^7 \times (10x - 4)$

Now substitute u. $= 24(5x^2 - 4x + 2)^7(10x - 4)$

Take out the common factor. $= 48(5x^2 - 4x + 2)^7(5x - 2)$

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b Write as a function of a function. Let $y = y[u(x)]$, where $y(u) = u^6$ and $u(x) = 2x - 1$

Find the derivatives. $\frac{dy}{du} = 6u^5$ and $\frac{du}{dx} = 2$

Write the chain rule. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Substitute the derivatives. $= 6u^5 \times 2$

Now substitute u. $= 12(2x - 1)^5$

Find the slope of the tangent. $m = \left.\frac{dy}{dx}\right|_{x=1} = 12 \times (2 \times 1 - 1)^5 = 12$

Check that the point is on the line. At $x = 1$, $y = (2 \times 1 - 1)^6 = 1$

Write the rule for the equation. $y - y_1 = m(x - x_1)$

Substitute values. $y - 1 = 12(x - 1)$

Simplify and write in standard form. $12x - y - 11 = 0$

State the answer. The equation of the tangent to $y = (2x - 1)^6$ at $(1, 1)$ is $12x - y - 11 = 0$.

After a while, you will be able to abbreviate the chain rule substitutions. Many problems will combine other formulas with the chain rule.

Example 5

Differentiate the following using the chain rule.

a $\frac{1}{(3x+1)^4}$ **b** $\sqrt{5x-4}$

Solution

a Write as a function of a function in index form. Let $y = \frac{1}{(3x+1)^4} = (3x + 1)^{-4} = u^{-4}$, where $u = 3x + 1$

Write the chain rule. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Substitute the derivatives. $= -4u^{-5} \times 3$

Substitute u. $= -12(3x + 1)^{-5}$

Write in the original form. $\frac{d}{dx}\left(\frac{1}{(3x+1)^4}\right) = -\frac{12}{(3x+1)^5}$

b Write as a function of a function. Let $y = \sqrt{5x-4} = u^{\frac{1}{2}}$, where $u = 5x - 4$

Write the chain rule. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Substitute the derivatives. $= \frac{1}{2}u^{-\frac{1}{2}} \times 5$

Substitute u. $= \frac{5}{2}(5x-4)^{-\frac{1}{2}}$

Write in the original form. $\dfrac{d}{dx}\left(\sqrt{5x-4}\right) = \dfrac{5}{2\sqrt{5x-4}}$

You may have to use the chain rule and the product or quotient rules in some problems.

WS

The chain rule

Example 6

Differentiate $2x^5(5x+3)^3$.

Solution

Write as a product.	Let $y = uv$, where $u = 2x^5$ and $v = (5x+3)^3$
Find the derivative of u.	$u' = 10x^4$
Write v as a function of a function.	Let $v(x) = p[q(x)]$, where $p(q) = q^3$ and $q(x) = 5x + 3$
Write the chain rule.	$v' = \dfrac{dp}{dx} = \dfrac{dp}{dq} \times \dfrac{dq}{dx}$
Substitute the derivatives.	$= 3q^2 \times 5$
Substitute q.	$= 15(5x+3)^2$
Write the product rule.	$y' = u'v + uv'$
Substitute the functions.	$= 10x^4(5x+3)^3 + 2x^5 \times 15(5x+3)^2$
Take out the common factor.	$= 10x^4(5x+3)^2[(5x+3) + 3x]$
Simplify and write the answer.	The derivative of $2x^5(5x+3)^3$ is $10x^4(5x+3)^2(8x+3)$.

Notice in Example **6** that if possible, it is usual to leave the derivative in the same form as the question. In this case it was factorised, so it has been left in factor form.

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EXERCISE 1.02 The chain rule

Concepts and techniques

Mixed differentiation problems

1 $m(x) = 3x + 3$, $g(x) = x^2$, $p(x) = \sqrt[3]{x}$ and $q(x) = x^2 + 4$

a What are $g(m)$, $m(x)$ and $g[m(x)]$?

b What are $m(g)$, $g(x)$ and $m[g(x)]$?

c Find $q[m(x)]$.

d Find $m[q(x)]$.

e What is $p[q(x)]$?

f What is $q[p(x)]$?

g Calculate the value of $m[p(8)]$.

h What is $p[m(8)]$?

i Find the value of $g[q(3)]$.

j Find the value of $q[g(3)]$.

2 Example 4 Differentiate the following.

a $(x + 3)^4$ b $(2x - 1)^3$ c $(5x^2 - 4)^7$

d $(8x + 3)^6$ e $(1 - x)^5$

3 Find the derivatives of the following.

a $3(5x + 9)^9$ b $2(x - 4)^2$ c $(2x^3 + 3x)^4$

d $(x^2 + 5x - 1)^8$ e $(x^6 - 2x^2 + 3)^6$

4 Example 5 Differentiate the following.

a $(3x - 1)^{\frac{1}{2}}$ b $(4 - x)^{-2}$ c $(x^2 - 9)^{-3}$

d $(5x + 4)^{\frac{1}{3}}$ e $(x^3 - 7x^2 + x)^{\frac{3}{4}}$

5 Find the derivatives of the following.

a $\sqrt{3x + 4}$ b $\dfrac{1}{5x - 2}$ c $\dfrac{1}{(x^2 + 1)^4}$

d $\sqrt[3]{(7 - 3x)^2}$ e $\dfrac{5}{\sqrt{4 + x}}$

6 Example 6 Find the derivatives of the following.

a $x^2(x + 1)^3$ b $4x(3x - 2)^5$ c $3x^4(4 - x)^3$

d $(x + 1)(2x + 5)^4$ e $(x^3 + 5x^2 - 3)(x^2 + 1)^5$

7 Differentiate the following.

a $\dfrac{(2x - 9)^3}{5x + 1}$ b $\dfrac{x - 1}{(7x + 2)^4}$ c $\dfrac{(3x + 4)^5}{(2x - 5)^3}$

d $\dfrac{3x + 1}{\sqrt{x + 1}}$ e $\dfrac{\sqrt{x - 1}}{2x - 3}$

8 a If $g(x) = (3x + 5)^3(2x - 1)^2$, find $g'(-2)$.

b If $f(t) = \dfrac{3t^2 - 2t + 1}{(4t - 5)^3}$, find $f'(3)$.

c If $p(x) = \sqrt[3]{(4x - 7)^5}$, find $p'(0.5)$.

d If $h(z) = (3z - 1)^4\left(\sqrt[3]{2z - 3}\right)^2$, find $h'(-1)$.

e If $m(x) = \dfrac{1}{(3x^2 - 2x - 4)^3}$, find $m'(1)$.

Reasoning and communication

9 Find any points on the curve $y = (2x - 3)^4$ where the gradient of the tangent is -8.

10 The volume of liquid in an underground tank being gravity-filled from a tanker is given by $V = (1500t + 17t^2)^3$, where the volume V is in litres and time t is in minutes.

a What is the volume of the tank after half an hour?

b What is the rate at which the tank is being filled after 5 minutes?

c What is the flow rate in the pipe after half an hour?

(Write all answers in scientific notation correct to 1 decimal place)

Getty Images/Justin Sullivan

11 Consider the function $q(x) = \sqrt{x-4}$.

a What is the domain of $q(x)$?

b Find the gradient of the tangent at $x = 13$

c Find the equation of the tangent at $x = 13$

d Find the gradient of the tangent at the point A, where $x = a$

e Find the gradient of the line from A to the origin.

f Hence find the point A on the curve such that the tangent passes through the origin.

12 Consider the function $y = \dfrac{1}{x+2} + 2$.

a Find the gradient of the tangent at $x = -2.5$

b Find the gradient of the tangent at $x = -1.5$

c What can you say about the tangents at $x = -2.5$ and $x = -1.5$?

d Find the gradient of the tangent at $x = -2.2$

e Find the gradient of the tangent at $x = -1.8$

f What can you say about the tangents at $x = -2.2$ and $x = -1.8$?

g Find the gradient of the tangent at $x = -3$

h Find the gradient of the tangent at $x = -1$

i What can you say about the tangents at $x = -3$ and $x = -1$?

j Write a general expression for the x-coordinates of the points on this curve that have parallel tangents.

13 A normal is perpendicular to a curve. Show that the normal to $y = (x + 2)^3$ at the point $(-3, -1)$ has y-intercept -2.

14 Show that the normals to the curve $y = 4x^2$ from points the same distance on either side of the y-axis intersect on the y-axis.

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1.03 THE DERIVATIVE OF EXPONENTIAL FUNCTIONS

You can find the derivative of $f(x) = a^x$ using first principles as follows.

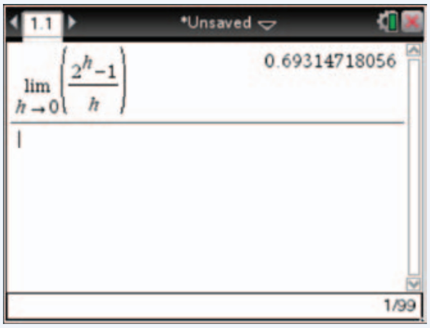
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x \times a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \end{aligned}$$

The derivative exists if $\lim\limits_{h\to 0} \frac{a^h - 1}{h}$ exists. In that case, the derivate of a^x is just a constant multiplied by the function itself!

INVESTIGATION Derivatives of a^x

1 You can estimate $\lim\limits_{h\to 0} \frac{a^h - 1}{h}$ for $a = 2$ using your CAS calculator as follows. Make sure that your calculator is set to give decimal answers (approximate calculation mode for the TI-Nspire Document settings).

TI-Nspire CAS



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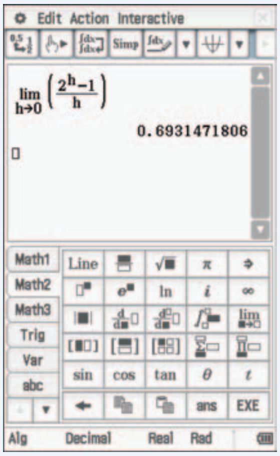
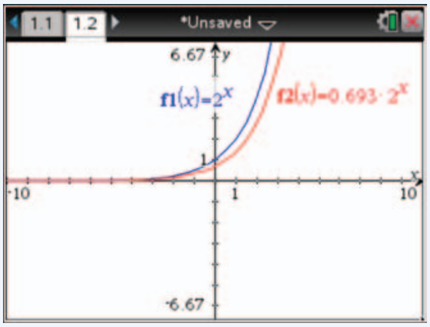


Estimate the values of $\lim\limits_{h\to 0}\dfrac{a^h-1}{h}$ for $a = 0.5, 0.8, 1.5, 2, 2.5, 3, 3.5, 4$ and 5.

2 Use these estimates to find approximations to the derivative $\dfrac{d}{dx}(a^x) = a^x \lim\limits_{h\to 0}\dfrac{a^h-1}{h}$ for the different values of a.

3 Using some of the results from the investigation, draw the graphs of exponential functions and their derivatives on your CAS calculator. For example, the graph of $f(x) = 2^x$ and its derivative $f'(x) \approx 0.693 \times 2^x$ looks like this.

TI-Nspire CAS



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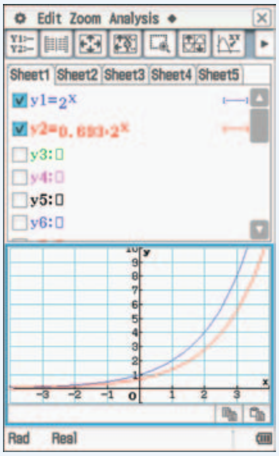

4 What happens to the value of $\lim\limits_{h\to 0}\dfrac{a^h-1}{h}$ as a increases?

From your investigation, you should see that there must be a number close to 3 for which $\lim\limits_{h\to 0}\dfrac{a^h-1}{h} = 1$. This number was alluded to by John Napier (1550–1617), the inventor of logarithms and was proven to be irrational by Leonard Euler (1707–1783). The number is now called e in honour of Euler and is called **Napier's number** or **Euler's number**. It is one of the most important numbers in mathematics.

IMPORTANT

The **exponential function** is the function $y = e^x$, where e is such that $\lim\limits_{h\to 0}\dfrac{e^h-1}{h} = 1$.

$e \approx 2.718\,281\,828\,459\,045\,235$, but cannot be written exactly.

The exponential function is its own derivative, so $\dfrac{d}{dx}(e^x) = e^x$.

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You can prove the derivative property as follows.

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} \\
&= \lim_{h\to 0}\frac{e^x \times e^h - e^x}{h} \\
&= \lim_{h\to 0}\frac{e^x(e^h-1)}{h} \\
&= e^x \lim_{h\to 0}\frac{(e^h-1)}{h} \\
&= e^x \qquad \text{because } \lim_{h\to 0}\frac{(e^h-1)}{h} = 1
\end{aligned}$$

You can find the value of e as accurately as you like using $e = \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots$

Example 7

a CAS Evaluate $3e^2 + 5$ to 2 decimal places.

b CAS Draw the graph of $y = e^x$.

c Find the exact equation of the tangent to the curve $y = e^x$ at the point $(2, e^2)$.

Solution

a Make sure that your calculator is set to decimal (approximate) calculation. Use e^x or e^n.

TI-Nspire CAS

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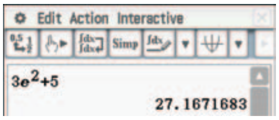
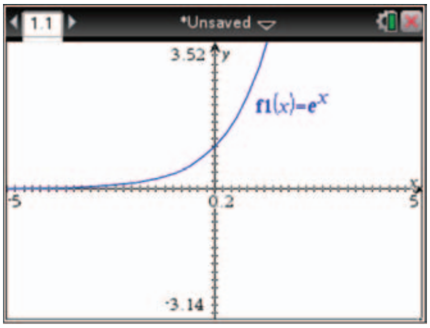


Write the answer.

$3e^2 + 5 \approx 27.17$ (correct to 2 decimal places).

b Use e^x or e^n for the graph.

TI-Nspire CAS



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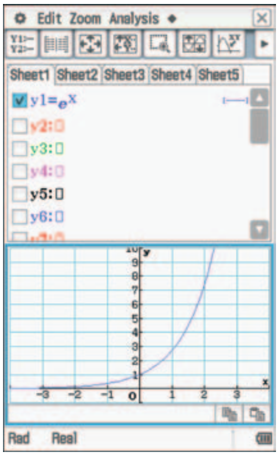

c Find the slope. $\frac{dy}{dx} = e^x$, so $m = e^2$

Write the equation of the line. $y - y_1 = m(x - x_1)$

Substitute values. $y - e^2 = e^2\,(x - 2)$

Express in standard form. The equation of the tangent is $e^2x - y - e^2 = 0$.

You can differentiate more complex functions involving the exponential function using some rules that have already been covered.

Example 8

Find the derivatives of the following.

a $4x^3 + 5e^x$

b $y = e^{x^3+x-2}$

c $f(x) = (3 + e^x)^7$

Solution

a Use the linear sum. $\frac{d}{dx}\left(4x^3 + 5e^x\right) = 4\frac{d}{dx}\left(x^3\right) + 5\frac{d}{dx}\left(e^x\right)$

Insert the derivatives and simplify. $= 12x^2 + 5e^x$

b Write as a function of a function. $y = e^u$, where $u(x) = x^3 + x - 2$

Write the chain rule. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Substitute the derivatives. $= e^u \times (3x^2 + 1)$

Substitute u. $= e^{x^3+x-2}\,(3x^2 + 1)$

c Write as a function of a function. $f(x) = v[u(x)]$, where $v(u) = u^7$ and $u(x) = 3 + e^x$

Write the derivatives. $v'(u) = 7u^6$ and $u'(x) = e^x$

Write the chain rule. $f'(x) = v'(u)u'(x)$

Substitute the derivatives. $= 7u^6 \times e^x$

Substitute u. $= 7e^x(3 + e^x)^6$

Example 9

Find the derivatives of the following.

a $x^2 e^x$ **b** $\dfrac{e^{2x}}{x}$

Solution

a Write as a product. Let $f(x) = u(x)v(x)$, where $u(x) = x^2$ and $v(x) = e^x$

Write the derivatives. $u'(x) = 2x$ and $v'(x) = e^x$

Write the product rule. $f'(x) = u'v + uv'$

Substitute in the functions. $= 2x \times e^x + x^2 \times e^x$

Factorise and write in normal order. $= xe^x(x+2)$

Write the answer. The derivative of $x^2 e^x$ is $xe^x(x+2)$.

b Write as a quotient. Let $f(x) = \dfrac{u(x)}{v(x)}$, where $u(x) = e^{2x}$ and $v(x) = x$

Write $u(x)$ as a function of a function. Let $u(x) = p[q(x)]$, where $p(q) = e^q$ and $q(x) = 2x$

Write the chain rule. $u'(x) = p'(q) \times q'(x)$

Substitute in the derivatives. $= e^q \times 2$

Substitute in q. $= 2e^{2x}$

Write the derivative of v. $v'(x) = 1$

Write the quotient rule. $f'(x) = \dfrac{u'v - uv'}{v^2}$

Substitute in the functions. $= \dfrac{2e^{2x} \times x - e^{2x} \times 1}{x^2}$

Factorise the numerator. $= \dfrac{e^{2x}(2x-1)}{x^2}$

Write the answer. The derivative of $\dfrac{e^{2x}}{x}$ is $\dfrac{e^{2x}(2x-1)}{x^2}$.

You will find after a while that you can skip many of the steps in the product, quotient and chain rules. However, it is still important to write enough so that your working is clear. If you are differentiating a function where you need to apply many rules, it is best to use 'dummy functions' like those in part **b** above to avoid errors. You will also find that you save time by doing this.

EXERCISE 1.03 The derivative of exponential functions

Concepts and techniques

1 Example 7 CAS Evaluate the following expressions, correct to 2 decimal places.

a $e^{1.5}$ b e^{-2} c $2e^{0.3}$ d $\frac{1}{e^3}$ e $-3e^{-3.1}$

2 CAS Draw these curves.

a $y=2e^x$ b $y=e^{-x}$ c $y=-e^x$

3 a Find the exact gradient of the tangent to the curve $y=e^x$ at the point $(1, e)$.

b Find the gradient of the tangent to the curve $y=e^x$ at the point where $x=0.58$, correct to 2 decimal places.

c Find the equation of the tangent to the curve $y=e^x$ at the point $\left(-1, \frac{1}{e}\right)$.

4 Example 8 Differentiate the following.

a $9e^x$ b $-e^x$ c e^x+x^2 d $2x^3-3x^2+5x-e^x$

e $(e^x+1)^3$ f $(5-e^x)^9$ g $(2e^x-3)^6$ h $(e^x+x)^4$

5 Find the derivative of

a e^{3x} b e^{2x-1} c $2e^{4x}$ d e^{x^2-1} e $e^{2x^5-3x^3+x-3}$

6 Example 9 Find the derivatives of the following.

a xe^x b $(2x+3)e^x$ c $5x^3e^x$ d $2x\,e^{3x}$ e $e^{2x}(x^2+x+2)$

7 Differentiate the following.

a $\frac{e^x}{x^2}$ b $\frac{e^{6x}}{3x}$ c $\frac{2e^{5x}}{5x^3}$ d $\frac{x-1}{e^x}$ e $\frac{e^x+1}{e^{2x}}$

8 a Find $g'(3)$ correct to 2 decimal places if $g(x)=\frac{e^x-4}{\sqrt{e^x+1}}$

b $y=e^{4x}(x^3-3x+5)$. Find $\left.\frac{dy}{dx}\right|_{x=-1}$

c If $f(x)=\frac{xe^{3x}+5}{x^2+e}$, find $f'(2)$ correct to 1 decimal place.

d $h(x)=5x^2e^{3x}+e^x$. Find $h'(2)$.

e If $y=5e^{x^2-x-6}$, find the values of y' when $x=1$ and $x=3$.

Reasoning and communication

9 Find the value of x such that the rate of change of xe^{2x-1} is $5e^3$.

10 $p(x)=e^{-kx}+3x$. For what value of k does the tangent to $p(x)$ at $x=2$ pass through the origin?

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1.04 APPLICATIONS OF THE EXPONENTIAL FUNCTION AND ITS DERIVATIVE

The exponential function and its derivative have many applications, as there are many natural phenomena that are modelled using simple exponential functions of the form $f(x) = Ae^{kx}$, where A and k are constants. In the case where x is time, you write $f(t) = Ae^{kt}$.

What is the value of $f(t)$ when $t = 0$? What does the graph look like when $k > 0$?

IMPORTANT

For $k > 0$, $Q = Ae^{kt}$ is used to model **exponential growth** over time t, where A is the initial quantity of the variable Q.

Exponential growth is used to model growth in populations, spread of disease and advances in technology, among other things. Since the exponential function is continuous, you need to round answers that have to be whole numbers.

Example 10

The increase in the number of gum trees with pink flowers in a region of South Australia was studied. The equation $N = 1200e^{0.07t}$ was given as a model for the increase, with N as the number of gum trees over time t years.

a How many trees were there at the beginning of the study?

b How many trees were there after 10 years?

c What was the rate of increase in the number of trees after 10 years?

Photo courtesy Margaret Grove

Solution

a Substitute $t = 0$ into $N = 1200e^{0.07t}$.	$N = 1200e^{0.07 \times 0}$ $= 1200$
State the result.	There were 1200 trees at the beginning.

b Substitute $t = 10$ into $N = 1200e^{0.07t}$.

$N = 1200e^{0.07 \times 10}$
$= 2416.503\ldots$

Round and state the result.

There were about 2417 trees after 10 years.

c The rate of change is the derivative.

$\frac{dN}{dt} = 1200 \times 0.07e^{0.07t}$
$= 84e^{0.07t}$

Substitute $t = 10$.

Rate of change $= 84e^{0.07 \times 10}$
$= 169.155\ldots$

Round and state the result.

The rate of increase after 10 years is about 169 trees/year.

IMPORTANT

For $k < 0$, $Q = Ae^{kt}$ is used to model **exponential decay** over time t, where A is the initial quantity of the variable Q.

Examples of exponential decay include radioactive decay, cooling, leakage and decline in the populations of endangered species.

Example 11

A metal cools down according to the formula $T = T_0e^{-0.1t}$, where T is the temperature difference with the surroundings in °C and t is in minutes. The initial temperature is 228°C and the room is at 20°C.

a Evaluate T_0, the initial temperature difference.

b Find the temperature difference after

i 5 minutes ii 20 minutes.

c What is the temperature after

i 5 minutes ii 20 minutes?

d Find the rate at which the metal is cooling after

i 5 minutes ii 20 minutes.

Solution

a Find the temperature difference with the surroundings.

$T_0 = 228°C - 20°C$
$= 208°C$

b i Substitute $t = 5$.

$T = 208e^{-0.1 \times 5}$
$= 126.158\ldots$

State the result.

The temperature difference is about 126.2°C.

ii Substitute $t = 20$.

$T = 208e^{-0.1 \times 20}$
$= 28.149\ldots$

State the result.

The temperature difference is about 28.1°C.

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c **i** Add the room temperature.	The temperature after 5 minutes is about 146.2°C.
ii Add the room temperature.	The temperature after 20 minutes is about 48.1°C.
d Find the derivative.	$\frac{dT}{dt} = 208 \times (-0.1e^{-0.1t}) = -20.8e^{-0.1t}$
i Substitute $t = 5$.	Rate of change $= -20.8e^{-0.5}$ $= -12.615\ldots$
The negative indicates cooling.	The metal is cooling at about 12.6°C /minute.
ii Substitute $t = 20$.	Rate of change $= -20.8e^{-2}$ $= -2.814\ldots$
State the result.	The metal is cooling at about 2.8°C /minute.

In the examples above, the rate of change of $Q(t) = Ae^{kt}$ is given by $Q(t) = Ake^{kt} = kQ(t)$. The reverse is also true, but the proof is beyond the scope of this course.

IMPORTANT

If $\frac{dQ}{dt} = kQ$, then $Q(t) = Ae^{kt}$, where A is the quantity when $t = 0$.
This means that if the rate of change of a quantity is proportional to the quantity, then it must be a simple exponential function of time.

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Example 12

A factory was producing 10 000 units per year and embarked on a productivity campaign by identifying problems in production. As a result, the production per year increased according to the equation $\frac{dP}{dt} = 0.02P$ units/year.

a Find an equation for the rate of production P.

b Find the number of units produced in the 7th year.

c Find the rate of increase after 7 years.

Solution

a Use $\frac{dP}{dt} = 0.02P$.	$\frac{dP}{dt} = 0.02P$ means $P = Ae^{0.02t}$, where A is a constant
The initial quantity is 10 000.	The equation is $P = 10\,000e^{0.02t}$
b Substitute $t = 7$.	$P = 10\,000e^{0.02 \times 7}$ $= 11\,502.737\ldots$
State the result.	About 11 503 units are produced in the 7th year.
c Find the rate of change.	$\frac{dP}{dt} = 10\,000 \times 0.02e^{0.02t}$ $= 200e^{0.02t}$
Substitute $t = 7$.	Rate of change $= 200e^{0.14}$ $= 230.054\ldots$
State the result.	Rate of increase of production after 7 years is 230 units/year.

In Example **12**, you could find the rate of increase of production after 7 years by substituting the quantity produced in the 7th year in $\frac{dP}{dt} = 0.02P = 0.02 \times 11\,502 \approx 230$.

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EXERCISE 1.04 Applications of the exponential function and its derivative

Reasoning and communication

1 Example 10 A disease is sweeping Australia according to the formula $N = 175e^{0.062t}$, where N is the number of new cases of the disease and t is the time in weeks.

a How many new cases are there

i initially? ii after 1 week? iii after 6 weeks? iv after 6 months?

b What is the rate at which new cases are found

i after 1 week? ii after 6 weeks? iii after 6 months?

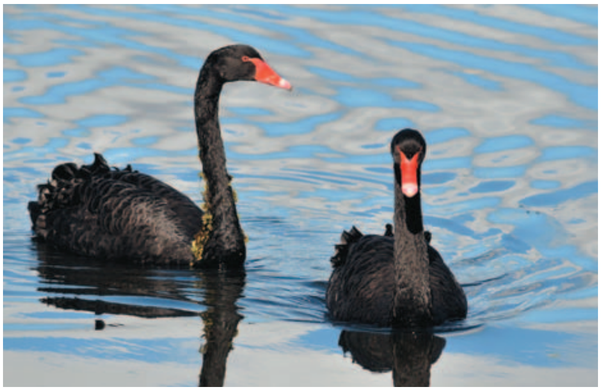
2 An advertising campaign helped increase sales of a product according to the formula $S = 180e^{0.12t}$, where S stands for the number of sales and t is the time in days.

a How many sales were made at the beginning of the advertising campaign?

b How many sales were made after 2 weeks of this campaign?

c What was the rate at which sales were made after 2 weeks?

d The advertising campaign finished after 6 weeks. What was the rate of sales at this time?



iStockphoto/Bbeale

3 A study of swans in an area of Western Australia showed that their numbers were gradually increasing, with the number of swans N over t months given by $N = 1100e^{0.025t}$.

a How many swans were there at the beginning of the study?

b How many swans were there after

i 5 months?

ii a year?

iii 3 years?

c At what rate was the number of swans increasing

i initially? ii after 5 months? iii after a year?

4 Example 11 The mass of uranium decays according to the formula $M = 200e^{-0.012t}$, where M is its mass in grams and t is the time in years.

a What is the initial mass of uranium?

b What is its mass after

i 5 years? ii 20 years? iii 100 years?

c At what rate is the mass decaying after

i 5 years? ii 20 years? iii 100 years?

5 The area of rainforests is declining in a region of Queensland with the area A hectares over time t years given by $A = 120\ 000e^{-0.033t}$.

a How many hectares of rainforest are there after

i 10 years? ii 25 years? iii 50 years?

b At what rate is the area of rainforest decreasing in this region after

i 2 years? ii 15 years? iii 40 years?

6 Example 12 The rate at which numbers N of bacteria are increasing over time t hours in a wound is proportional to the number of bacteria present. This can be modelled by the formula $\frac{dN}{dt} = 0.29N$.

a If there are initially 90 000 bacteria present, find a formula that models the number of bacteria present.

b How many bacteria are present after 6 hours?

c At what rate are the numbers increasing after 6 hours?

d What is the rate at which bacteria are increasing after 10 hours?

7 The average annual rainfall R is decreasing over time t years in a region of Tasmania according to the formula $\frac{dR}{dt} = -0.008R$. The first average annual rainfall measured in this region was 43 cm.

a Find a formula for the average annual rainfall in this region.

b Find the average annual rainfall after

i 10 years ii 30 years iii 100 years.

c What is the rate at which the rainfall is decreasing after

i 10 years ii 30 years iii 100 years?

8 The population P of a city after t years is given by the formula $P = P_0e^{0.024t}$.

a What is the initial population?

b By what percentage does the population increase after 6 years?

c Write a formula for the rate at which the population changes in terms of P.

9 The formula for the decay of a radioactive substance over t years is given by $Q = Q_0e^{-0.07t}$.

a What percentage of the original quantity is left after

i 2 years? ii 10 years? iii 20 years?

b The half-life is the time taken to decay to half of the original quantity. What is the approximate half-life of this substance?

10 a Find the equation of the tangent to the curve $y = e^{2x}$ at the point $M(2, e^4)$.

b Find the x-intercept N of this tangent.

c Find the exact area of triangle MNP, where P is the point on the x-axis directly below M.

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11 A project is introduced to increase the number of water birds in Macquarie Marshes. The model for the population P of water birds over time t months is given by the formula $P = 100 + 2e^{0.3t}$.

a How many water birds are there at the start of the project?

b What is the population after

i 6 months? ii 2 years?

c Find a formula for the rate at which the population increases over time $\left(\frac{dP}{dt}\right)$.

d At what rate is the water bird population increasing after

i 6 months? ii 2 years?

Photo courtesy Margaret Grove

12 The number N of bacteria in a sample is given by the formula $N = N_0\, e^{1.2t}$, where t is the time in hours.

a If there are initially 30 000 bacteria found in the sample, find the value of N_0.

b Find the number of bacteria in the sample after 5 hours.

c Find the rate at which the number of bacteria is increasing after

i 5 hours ii 12 hours iii 1 day.

1.05 THE DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

You can use your CAS calculator or a computer to estimate the derivatives of sin (x) and cos (x) for different values of x. In the case of derivatives, you must work in radians, so make sure that your calculator is set to radians.

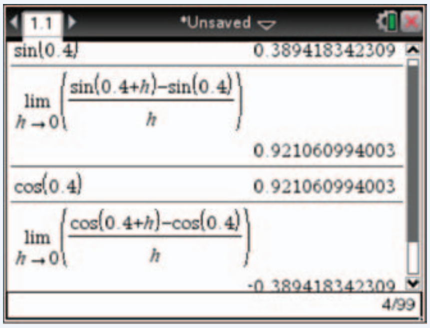
INVESTIGATION Estimating the derivatives of sin (*x*) and cos (*x*)

The derivative of sin (x) is given by $\lim\limits_{h \to 0} \dfrac{\sin(x+h)-\sin(x)}{h}$,

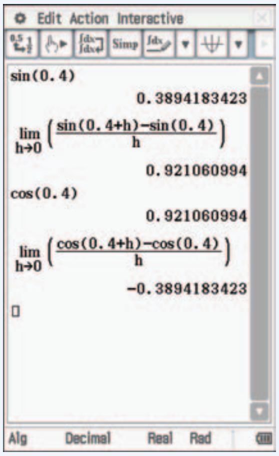
so at $x = 0.4$, the derivative is $\lim\limits_{h \to 0} \dfrac{\sin(0.4+h)-\sin(0.4)}{h}$.

The derivative of cos (x) is worked out in the same way. Do this on your CAS calculator as follows, with your calculator set to approximate/decimal calculation.

TI-Nspire CAS

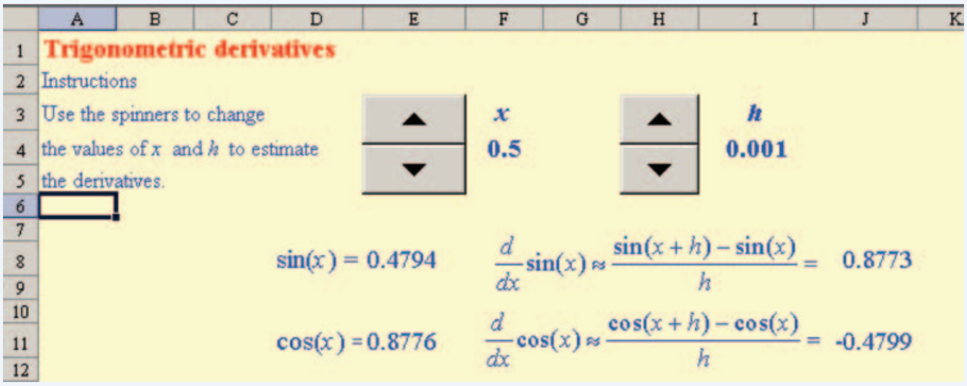


ClassPad



You can also estimate the derivatives using the Spreadsheet 'Trigonometric derivatives' from the website.

Trigonometric derivatives



What do you find about the estimates of the derivatives?

What happens as h is made smaller?

From the investigation, you can see that it appears that the derivative of sin (x) is cos (x) and that the derivative of cos (x) is −sin (x).

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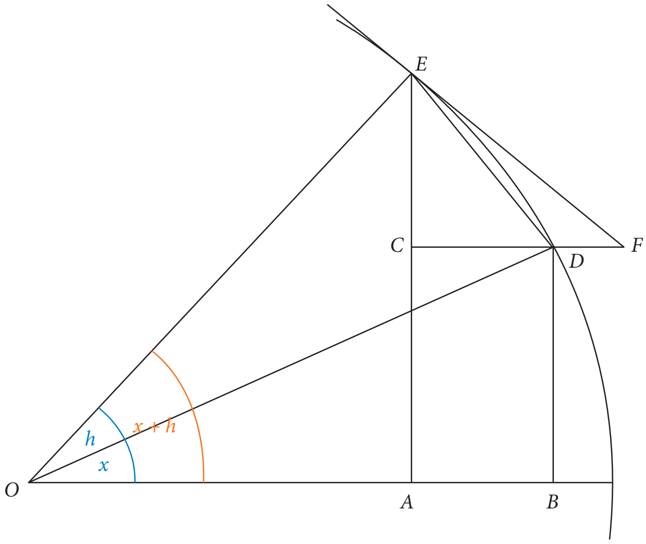
The derivative of sin (x) is given by $\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$. What does $\frac{\sin(x+h)-\sin(x)}{h}$ look like on a unit circle?

The diagram shows part of the unit circle, with angles x and $x + h$ shown.



Perpendiculars AE and BD are drawn from line OAB to radii OE and OD. Line CDF is drawn at 90° to AE to the tangent EF.

Line ED is also drawn.

From the definition of sine, $\sin(x+h) = EA$ and $\sin(x) = DB = CA$.

Thus, $\sin(x+h) - \sin(x) = EA - CA = EC$.

From the definition of a radian, the length of the arc ED is h, so

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{EC}{\text{arc } ED} \qquad [1]$$

Now, in ΔECF, $\angle FEC = x + h$, so

$$\cos(x+h) = \frac{EC}{EF} \qquad [2]$$

Also, ΔOED is isosceles, so $\angle OED = \frac{\pi - h}{2}$. Now $\angle OEA = \frac{\pi}{2} - (x+h)$, so

$\angle CED = \angle OED - \angle OEA$

$$= \frac{\pi - h}{2} - \left[\frac{\pi}{2} - (x+h)\right].$$

$$= x + \frac{h}{2}$$

Thus in ΔCED,

$$\cos\left(x + \frac{h}{2}\right) = \frac{EC}{ED} \qquad [3]$$

From the diagram, it is clear that $ED <$ arc $ED < EF$, so

$$\frac{EC}{EF} < \frac{EC}{\text{arc } EF} < \frac{EC}{ED}.$$

It follows from [1], [2] and [3] that

$$\cos(x+h) < \frac{\sin(x+h)-\sin(x)}{h} < \cos\left(x+\frac{h}{2}\right).$$

In the limit, as $h \to 0$,

$$\lim_{h\to 0}\cos(x+h) \le \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \le \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right).$$

But

$$\lim_{h\to 0}\cos(x+h) = \cos(x),$$

$$\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right) = \cos(x) \quad \text{and}$$

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \frac{d}{dx}\sin(x)$$

Thus $\cos(x) \le \frac{d}{dx}\sin(x) \le \cos(x)$, so $\frac{d}{dx}\sin(x) = \cos(x)$.

The diagram is drawn for the first quartile, but it can be drawn in other quartiles and the logic of the proof would be unchanged, so the proof is valid for any value of x.

IMPORTANT

The **derivative of sin (x)** is given by $\frac{d}{dx}\sin(x) = \cos(x)$.

Example 13

a Differentiate

i $3\sin(x)$ **ii** $f(x) = \sin(5x)$ **iii** $y = x\sin(x)$ **iv** $g(x) = [x^3 + \sin(x)]^5$

b Find the exact gradient of the tangent to the curve $y = \sin(2x)$ at the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$.

Solution

a i Use the linear product rule.

$$\frac{d}{dx}[3\sin(x)] = 3\frac{d}{dx}[\sin(x)]$$
$$= 3\cos(x)$$

ii Use the chain rule with $u = 5x$.

$$f(x) = 5 \times \cos(5x)$$
$$= 5\cos(5x)$$

iii Write as a product of functions.

Let $y = uv$, where $u = x$ and $v = \sin(x)$

Write the product rule.

$$y' = u'v + uv'$$

Substitute the functions.

$$= 1 \times \sin(x) + x\cos(x)$$

Simplify.

$$= \sin(x) + x\cos(x)$$

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iv Write as a function of a function.

Let $g(x) = v(u(x))$, where $v(u) = u^5$ and $u(x) = x^3 + \sin(x)$

Write the chain rule.

$g'(x) = v'(u)u'(x)$

Substitute the derivatives.

$= 5u^4 \times [3x^2 + \cos(x)]$

Substitute u.

$= 5[x^3 + \sin(x)]^4\,[3x^2 + \cos(x)]$

b Find the derivative.

$$\frac{dy}{dx} = 2 \times \cos(2x)$$
$$= 2\cos(2x)$$

Substitute $x = \frac{\pi}{6}$.

$$= 2\cos\left(2 \times \frac{\pi}{6}\right)$$
$$= 2 \times 0.5$$
$$= 1$$

State the result.

The gradient of the tangent is 1.

Remember that $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$. You can use these to find the derivative of $\cos(x)$.

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right)$$
$$= -1 \times \cos\left(\frac{\pi}{2} - x\right)$$
$$= -\sin(x)$$

IMPORTANT

The **derivative of cos (x)** is given by $\frac{d}{dx}\cos(x) = -\sin(x)$

Example 14

Differentiate

a $y = 1 - 8\cos(x)$

b $\cos(\pi x)$

c $g(x) = \dfrac{\cos(4x)}{x}$

d $h(x) = e^{\cos(x)}$

Solution

a Use the linear product rule.

$$\frac{dy}{dx} = 0 - [-8\sin(x)] = 8\sin(x)$$

b Use the chain rule with $u = \pi x$

$$\frac{d}{dx}\cos(\pi x) = -\sin(\pi x) \times \pi = -\pi\sin(\pi x)$$

c Write as a quotient of functions. $g(x) = \dfrac{\cos(4x)}{x}$, where $u(x) = \cos(4x)$ and $v(x) = x$

Write the quotient rule. $g'(x) = \dfrac{u'v - uv'}{v^2}$

Substitute functions. $= \dfrac{-4\sin(4x) \times x - \cos(4x) \times 1}{x^2}$

Simplify. $= \dfrac{-4x\sin(4x) - \cos(4x)}{x^2}$

d Write as a function of a function. Let $h(x) = v[u(x)]$, where $v(u) = e^u$ and $u(x) = \cos(x)$

Write the chain rule. $g'(x) = v'(u)u'(x)$

Substitute the derivatives. $= e^u \times -\sin(x)$

Substitute u. $= -e^{\cos(x)} \sin(x)$

You can find the derivative of $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ using the quotient rule.

Let $\tan(x) = \dfrac{u(x)}{v(x)}$.

$$\text{Then } \frac{d}{dx}\tan(x) = \frac{u'v - uv'}{v^2}$$

$$= \frac{\cos(x) \times \cos(x) - \sin(x) \times [-\sin(x)]}{[\cos(x)]^2}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)}, \text{ since } \sin^2(x) + \cos^2(x) = 1.$$

IMPORTANT

The **derivative of tan (x)** is given by $\dfrac{d}{dx}[\tan(x)] = \dfrac{1}{\cos^2(x)}$.

Example 15

a Differentiate

i $y = 7\tan(x)$ **ii** $g(x) = \tan(4x)$

iii $h(x) = 3x^2 \tan(x)$ **iv** $\tan(1 + e^x)$

b Find the exact gradient of the tangent to the curve $y = \tan(x)$ at the point where $x = \frac{\pi}{6}$.

Solution

a i Use the linear factor rule.

$$\frac{dy}{dx} = 7 \times \frac{1}{\cos^2(x)} = \frac{7}{\cos^2(x)}$$

ii Use the chain rule with $u = 4x$

$$g'(x) = \frac{1}{\cos^2(4x)} \times 4 = \frac{4}{\cos^2(4x)}$$

iii Write as a product.

$h(x) = uv$, where $u(x) = 3x^2$ and $v(x) = \tan(x)$

Write the product rule.

$g' = u'v + uv'$

Substitute functions.

$$= 6x \times \tan(x) + 3x^2 \times \frac{1}{\cos^2(u)}$$

Simplify.

$$= 6x\tan(x) + \frac{3x^2}{\cos^2(x)}$$

iv Write as a function of a function.

Let $\tan(1 + e^x) = v[u(x)]$, where $v(u) = \tan(u)$ and $u(x) = 1 + e^x$

Write the chain rule.

$$\frac{d}{dx}\tan(1 + e^x) = v'(u)u'(x)$$

Substitute the derivatives.

$$= \frac{1}{\cos^2(u)} \times e^x$$

Substitute u.

$$= \frac{e^x}{\cos^2(1 + e^x)}$$

b Find the derivative.

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

Substitute $x = \frac{\pi}{6}$.

$$= \frac{1}{\cos^2\left(\frac{\pi}{6}\right)}$$

Use $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{4}{3}$$

Write the answer.

The exact gradient is $1\frac{1}{3}$.

EXERCISE 1.05 The derivatives of trigonometric functions

Concepts and techniques

WS Differentiating trigonometric functions

1 Example 13 Find the derivatives of the following.

a $x + \sin(x)$ b $6\sin(x)$ c $\sin(6x)$

d $\sin(x^2 - 3)$ e $4\sin\left(\frac{x}{3}\right)$

2 Differentiate the following.

a $[\sin(x) + 9]^6$ b $x\sin(x)$ c $\sin(e^x)$

d $\frac{\sin(2x)}{5x^2}$ e $e^{\sin(x)}$

3 Find the gradient of the tangent to the following at the given points.

a $y = \sin(x)$ at the point where $x = \frac{\pi}{3}$

b $y = 3\sin(4x)$ at the point $\left(\frac{\pi}{16}, \frac{3}{\sqrt{2}}\right)$

c $f(x) = \sin(x)$ at the point $\left(\frac{3\pi}{2}, -1\right)$

d $y = x + \sin(x)$ at the point where $x = \frac{\pi}{6}$

e $y = \frac{\sin(x)}{x}$ at the point where $x = \frac{\pi}{2}$

4 Example 14 Differentiate the following.

a $2 + \cos(x)$ b $3\cos(x)$ c $\cos(5x)$

d $\cos(3x^2 + 1)$ e $2\cos\left(\frac{x}{2}\right)$

5 Find the derivatives of the following.

a $[4x + \cos(x)]^3$ b $x\cos(x)$ c $\cos(e^{3x})$

d $\frac{\cos(x)}{3x}$ e $\cos[\sin(x)]$

6 Example 15 Differentiate the following.

a $\tan(x)$ b $x + 6\tan(x)$ c $\tan(9x)$

d $3\tan(4x)$ e $[\tan(x) - 1]^5$

7 Differentiate the following.

a $x^2\tan(x)$ b $\frac{\tan(2x)}{x}$ c $e^{\tan(x)}$

d $\tan[\cos(x)]$ e $\frac{\tan(x)}{e^x}$

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8 Find the exact values of the derivatives of the following functions at the given value of the variable.

a $e^{2x}\sin(x)$ at $x = 0.5$

b $x^2 \tan(x)$ at $x = \frac{\pi}{4}$

c $\cos^2(x)$ at $x = \frac{\pi}{3}$

d $\frac{\cos\left(e^x\right)}{e^x}$ at $x = 1$

e $x^3 \cos(x^2)$ at $x = e$

9 CAS Find each of the following correct to three decimal places.

a The value of $p'\left(\frac{\pi}{6}\right)$, where $p(x) = x^2 \sin(x) - x\cos(x)$

b The value of $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}}$ for $y = \sqrt{\cos(x)}$

Reasoning and communication

10 Find all x-values for which the gradient of the tangent to the curve $y = \tan(x)$ is 2.

11 The equation of the displacement x cm of a particle is given by $x = 2\sin(3t)$, where the time t is in seconds. The particle is at the centre of the displacement path when $x = 0$.

a What is the particle's greatest displacement from the centre?

b At what times is the particle at the centre?

c At what rate is the particle moving after $\frac{\pi}{6}$ seconds?

d What are the first 3 times that the rate of movement is ±3 cm/s?

12 The velocity V of a guitar string moving over time t seconds is given by $V = \sin(2t) + 3t + 1$ mm/s.

a What is its velocity after $\frac{\pi}{12}$ seconds, correct to one decimal place?

b Given that acceleration is the rate of change of velocity, find the acceleration of the string after $\frac{\pi}{4}$ seconds.

c Show that the acceleration of the string is never 0.

1.06 APPLICATIONS OF TRIGONOMETRIC FUNCTIONS AND THEIR DERIVATIVES

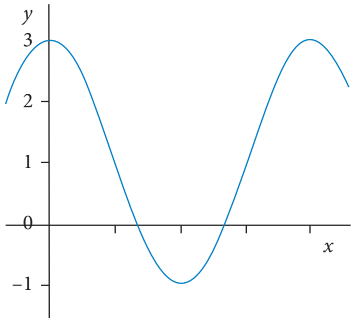
The slope m of a line is related to the angle θ it makes with the positive direction of the x-axis by $m = \tan(\theta)$; the slope of a curve at any point is given by its derivative, which is the slope of the tangent. The direction of the tangent is taken as direction of the line.

Example 16

Find the angle between the x-axis and the curve $y = 2\cos(x) + 1$ at its second point of intersection with the x-axis ($x > 0$).

Solution

Sketch the graph.



Write the equation for the intersection.	$0 = 2\cos(x) + 1$
Solve to find x.	$\cos(x) = -\frac{1}{2}$, so $x = \frac{2\pi}{3}$
Use your knowledge of the graph.	The second intersection is at $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$
Find the slope of the curve.	$m = \frac{dy}{dx}$ $= -2\sin(x)$
Substitute $x = \frac{4\pi}{3}$.	$= -2\sin\left(\frac{4\pi}{3}\right)$ $= -2 \times \left(-\frac{\sqrt{3}}{2}\right)$ $= \sqrt{3}$
Write the equation for the inclination.	$m = \tan(\theta)$
Substitute the slope.	$\sqrt{3} = \tan(\theta)$
Find the angle.	$\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$
From the diagram, the angle is acute.	The angle at the second intersection is $\frac{\pi}{3}$.

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Example 17

Find the equation of the tangent to the curve $y = \cos(x)$ at the point where $x = \frac{\pi}{4}$.

Solution

Find the y-coordinate.	$y = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
Differentiate $y = \cos(x)$.	$\frac{dy}{dx} = -\sin(x)$
Substitute $x = \frac{\pi}{4}$.	$m = \frac{dy}{dx} = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
Write the equation of the tangent.	$y - y_1 = m(x - x_1)$
Substitute $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and m.	$y - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$
Simplify.	$4\sqrt{2}y - 4 = -4x + \pi$
Write the result.	The equation of the tangent is $4x + 4\sqrt{2}y - 4 - \pi = 0$.

Remember that velocity is the derivative of displacement and acceleration is the derivative of velocity.

Example 18

A spring moves so that its end is x cm from the point P at time t seconds, where $x = 2\sin(4t)$.

a Find an equation for the velocity of the spring.

b What is the initial velocity of the spring?

c When is the velocity first equal to zero?

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Solution

a Differentiate to find v.	$v = 2 \times 4\cos(4t)$ $= 8\cos(4t)$
b Substitute $t = 0$.	$= 8\cos(4 \times 0)$ $= 8$
State the result.	The initial velocity is 8 cm/s.

c Substitute $v = 0$ into the velocity equation.	$0 = 8\cos(4t)$
Solve to find t.	$4t = \cos^{-1}(0)$
Solve the equation.	$4t = \frac{\pi}{2}$, so $t = \frac{\pi}{8} = 0.3926\ldots$
State the result.	The velocity is first 0 at about 0.39 seconds.

EXERCISE 1.06 Applications of trigonometric functions and their derivatives

Reasoning and communication

1 Example 16 Find the angle between the x-axis and the curve $y = 3\sin(x) - 2$ at its first point of intersection with the x-axis $(x > 0)$.

2 Find the angle between the curves $y = \sin(x)$ and $y = \cos(x)$ at the first point at which they intersect in the domain $0 \le x \le 5$.

3 Find the angle between the curves $y = \sin(3x)$ and $y = \cos(3x)$ at the first point at which they intersect in the domain $0 \le x \le 5$.

4 A cam is a non-circular shaft used to work valves and other parts in machines. A particular cam has a profile modelled by the curves $y = 3\cos\left(\frac{\pi x}{4}\right)$ and $y = -3\cos\left(\frac{\pi x}{4}\right)$ for $-2 \le x \le 2$, where x and y are both in centimetres. The 'lift' of the cam is the difference in height of the top of the cam from the centre as it rotates.

a Sketch the shape of the cam.

b What is the 'lift' of the cam?

c What is the angle at the 'points' of the profile?

5 A moving sand dune has a crescent shape such that the edges of the dune are modelled by the curves $y = 80\sin\left(\frac{\pi x}{80}\right)$ and $y = 30\sin\left(\frac{\pi x}{80}\right)$ from $x = 0$ to $x = 80$, where x and y are both in metres.

a Sketch the shape of the dune.

b What is the maximum thickness of the dune?

c Find the angle of the 'points' of the dune.

6 Example 17 Find the equation of the tangents to the following curves at the stated points.

a $y = \sin(x)$ at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$

b $y = -2\sin\left(\frac{x}{2}\right)$ at the point where $x = \frac{\pi}{2}$

c $y = \cos(x)$ at the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

d $y = \sin(2x)$ at the point $\left(\frac{\pi}{12}, \frac{1}{2}\right)$

e $y = \tan(3x)$ at the point $\left(\frac{\pi}{12}, 1\right)$

7 Example 18 A particle moves so that its displacement is $x = 3\cos\left(\frac{t}{2}\right)$, where x is in centimetres and t is in seconds.

a Find an equation for the velocity of the particle.

b Find an equation for the acceleration of the particle.

c Find the times when the particle is at $x = 0$.

d What is the velocity and acceleration at these times?

e Find the times at which the particle has the greatest acceleration.

8 The speed of a pendulum bob is given by $v = 1.26\sin(2\pi t)$, where v is in metres/second and t is in seconds.

a Find an expression for the acceleration of the bob.

b Find the velocity and acceleration after 5 seconds.

c Find the acceleration at the times when the velocity is -1.26 m/s.

9 The tidal variation in water level in a particular bay can be modelled as $d = d_0 - 0.9\cos(0.503t)$, where d is the depth of water in metres at a particular point, t is the time in hours after low tide and d_0 is the depth of water at the same point at half-tide. Two hours after low tide the water reaches a mudflat sloped upwards at an angle of 2°. At approximately what speed does it then move horizontally across the mudflat towards the shoreline?

10 a If the displacement of a particle is given by $x = 2\sin(3t)$, show that its acceleration is given by $a = -9x$

b Given that displacement $x = a\cos(nt)$, show that its acceleration is given by $a = -n^2x$

1

CHAPTER SUMMARY

DERIVATIVES, EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

- The derivative of $f(x)$ is defined as $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
- The derivative of a constant is zero: $\frac{d}{dx}(k)=0$ for $k \in \boldsymbol{R}$
- The derivative of a linear product is given by $\frac{d}{dx}[kf(x)]=kf'(x)$ for $k \in \boldsymbol{R}$
- The derivative of a sum of functions is given by $\frac{d}{dx}[f(x)+g(x)]=f'(x)+g'(x)$
- The derivative of a difference of functions is given by $\frac{d}{dx}[f(x)-g(x)]=f'(x)-g'(x)$
- The derivative of a linear sum is given by $\frac{d}{dx}[af(x)+bg(x)]=af'(x)+bg'(x)$ for $a, b \in \boldsymbol{R}$
- The derivative $\frac{dy}{dx}$ is the **gradient function** of the curve $y=f(x)$
- A linear function through (x_1, y_1) with gradient m is given by $y-y_1=m(x-x_1)$
- The **product rule** for functions u and v can be written as $\frac{d}{dx}(uv)=u'v+uv'$
- The **quotient rule** for functions u and v can be written as $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}$
- Rules for negative and fractional indices: $x^{-n}=\frac{1}{x^n}$, $x^{\frac{1}{n}}=\sqrt[n]{x}$ and $x^{\frac{m}{n}}=\sqrt[n]{x^m}=\left(\sqrt[n]{x}\right)^m$
- The derivative of a power is given by $\frac{d}{dx}(x^n)=nx^{n-1}$ for $n \in \boldsymbol{R}$
- The **chain rule** states that $\frac{d}{dx}[f(g(x)]=f'(g)g'(x)$, where $g=g(x)$. It can also be written as if $f(x)=v[u(x)]$, then $f'(x)=v'(u)u(x)$, or $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$
- The **exponential function** is the function $y=e^x$, where e is such that $\lim_{h\to 0}\frac{e^h-1}{h}=1$
- $e \approx 2.718\ 281\ 828\ 459\ 045\ 235$, but cannot be written exactly.
- The exponential function is its own derivative, so $\frac{d}{dx}(e^x)=e^x$
- For $k>0$, $Q=Ae^{kt}$ is used to model **exponential growth** over time t, where A is the initial quantity of the variable Q
- For $k<0$, $Q=Ae^{kt}$ is used to model **exponential decay** over time t, where A is the initial quantity of the variable Q
- If the rate of change of a function is proportional to the functions, then it must be a simple exponential function. If $\frac{dQ}{dt}=kQ$, then $Q(t)=Ae^{kt}$, where A is the initial quantity
- The **derivative of sin (x)** is given by $\frac{d}{dx}\sin(x)=\cos(x)$
- The **derivative of cos (x)** is given by $\frac{d}{dx}\cos(x)=-\sin(x)$
- The **derivative of tan (x)** is given by $\frac{d}{dx}\tan(x)=\frac{1}{\cos^2(x)}$

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CHAPTER REVIEW
DERIVATIVES, EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

Multiple choice

1 Example 1 The derivative of $(3x^2 - 5x + 2)(x^4 + 3x^3 + x - 6)$ is:

A $24x^4 + 34x^3 - 39x^2 - 5$

B $18x^5 + 20x^4 - 52x^3 + 27x^2 - 46x + 32$

C $18x^5 - 7x^4 + 11x^3 - 21x^2 - 34x + 32$

D $18x^5 + 22x^4 - 46x^3 + 9x^2 - 14x + 8$

E $24x^4 + 34x^3 - 45x^2 + 6x - 5$

2 Example 2 The derivative of $\dfrac{2x+1}{3x-2}$ is

A $\dfrac{-1}{(3x-2)^2}$ B $\dfrac{-7}{(3x-2)^2}$ C $\dfrac{12x+7}{(3x-2)^2}$

D $\dfrac{12x-1}{(3x-2)^2}$ E $\dfrac{1}{(3x-2)^2}$

3 Example 3 The derivative of $\dfrac{1}{x^4}$ is

A $-\dfrac{1}{4x^3}$ B $-\dfrac{4}{x^3}$ C $\dfrac{4}{x^5}$ D $-\dfrac{4}{x^5}$ E $-\dfrac{1}{4x^5}$

4 Example 4 If $g(x) = (x^3 - 3x + 1)^3$, then $g'(-2) =$

A 0 B 3 C 27 D 243 E 729

5 Example 8 The gradient of the tangent to the curve $y = e^{2x}$ at the point where $x = 1.5$ is

A 7.4 B 9 C 14.8 D 20.1 E 40.2

6 Example 15 Given $y = 4\sin(e^x) + 2$, $\left.\dfrac{dy}{dx}\right|_{x=2} \approx$

A −9.3 B −7.58 C 11.3 D 13.3 E 31.3

7 Example 16 The equation of the tangent to the curve $y = \cos(3x)$ at the point where $x = \dfrac{\pi}{9}$ is

A $9x + 6y - 3\sqrt{3} - \pi = 0$

B $9\sqrt{3}x + 6y - 3 - \sqrt{3}\pi = 0$

C $9\sqrt{3}x + 18y - 9\sqrt{3} - \sqrt{3}\pi = 0$

D $9\sqrt{3}x - 6y + 3 - \sqrt{3}\pi = 0$

E $9\sqrt{3}x + 18y - 9 - \sqrt{3}\pi = 0$

Short answer

8 Example 1 Differentiate

a $x^5(3x^2 + 2x - 5)$

b $(x^2 + 1)(x^3 - 4x - 1)$

9 Example 2 Find the derivative of

a $\dfrac{5x^3}{2x+1}$

b $\dfrac{x^2 + x - 2}{4x - 3}$

10 Example 3 Find the derivative of

a x^{-6} b $x^{\frac{1}{7}}$ c $\sqrt{x^3}$ d $\dfrac{9}{x^2}$

11 Example 4 Differentiate

a $(2x-7)^5$ b $3(2x^3+x^2-3)^4$

12 Example 5 Differentiate

a $(x^5+1)^{-3}$ b $\sqrt[3]{x-1}$

13 Example 6 Find the derivative of

a $3x^2(x+2)^8$ b $\dfrac{3x^2}{(5x-1)^4}$

14 Example 7 Find

a the gradient of the tangent to the curve $y=e^x$ at the point $(3, e^3)$

b the equation of the tangent to the curve $y=3e^x$ at the point where $x=-2$

15 Example 8 Differentiate

a x^2+2e^x b e^{6x} c $(e^x-3)^9$ d $2e^{4x+1}$ e e^x+e^{-x}

16 Example 9 Differentiate

a $(4x+3)e^{2x}$ b $\dfrac{e^{3x}}{x-4}$

17 Example 10 The number N of apps sold by a company over t months is given by $N=1000e^{0.24t}$. Find

a the number of apps sold initially

b the number of apps sold after

i 6 months ii 2 years

c the rate at which apps are sold after

i 6 months ii 2 years

18 Example 11 The temperature $T°$ of a chemical that cools down over t minutes is given by $T=75e^{-0.15t}$. Find

a the temperature after

i 2 minutes ii 5 minutes.

b the rate at which the temperature changes after

i 2 minutes ii 5 minutes.

19 Example 12 The quantity of salt increases in an evaporating rock pool according to the equation $\dfrac{dQ}{dt}=0.04Q$. Initially the pool has 375 g of salt.

a Find an equation for the quantity Q of salt in the pool in terms of t, where t is in weeks.

b Find the amount of salt in the pool after 6 weeks

c Find the rate at which salt is increasing after 6 weeks.

20 Example 13 Differentiate

a $3\sin(x)+1$ b $\sin(5x)$ c $x\sin(2x)$

d $[3x-\sin(x)]^7$ e $\sin(x^3+1)$

21 Example 14 Find the derivative of

a $\cos\left(\frac{x}{3}\right)$

b $e^{x}\cos(x)$

c $[2+\cos(3x)]^{5}$

d $\cos(\pi x)$

e $\cos^{2}(x)$

22 Example 15 Differentiate

a $6\tan(x)$

b $\tan(3x)$

c $\tan\left(\frac{\pi x}{5}\right)$

d $x^{3}\tan(2x)$

e $\frac{\tan(x)}{x}$

Application

23 Show that the derivative of $Q = Q_0e^{kt}$ is $\frac{dQ}{dt} = kQ$

24 a Find the equation of the tangent to the curve $y = \tan(x)$ at the point $\left(\frac{\pi}{4}, 1\right)$.

b Find points A and B where this tangent meets the x- and y-axes.

c Find the area of triangle OAB, where O is the origin.

25 A pendulum moves so that its displacement x cm over time t seconds is given by $x = 6\sin(t)$.

a Find the displacement after 5 seconds.

b Find the times when the pendulum is at $x = 0$.

c Find the velocity of the pendulum after 3 seconds.

26 The profile of a skateboard 'jump' has been designed so that it follows the curve

$y = 2\cos\left(\frac{\pi x}{4}\right) + 2$ for $0 \le x \le 6$, where y and x are both in metres.

a Sketch the jump.

b What is the take-off angle at the end of the jump (in degrees)?

Practice quiz